Conversion of Frequency Stability Measures from the Time-domain to the Frequency-domain, vice-versa and Power-law Spectral Densities

By

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In general, a frequency standard will have a nominally constant frequency output, v_o. The lack of constancy, the frequency instability, can be measured in the frequency-domain or the time-domain. The conversion from frequency-domain measures to time-domain measures utilizes an integral equation. For example, Dr. Leonard S. Cutler originally (1972) derived the following for the time-domain variance known as the Allan variance:

$$\sigma_y^2(\tau) = 2 \int_0^\infty S_y(f) \frac{sin^4(\pi f \tau)}{(\pi f \tau)^2} \, df$$

In practice, the integral over the power spectral density (PSD), $S_y(f)$ of the normalized frequency, y, is from zero (0) to the high-frequency cutoff (f_h) of the measurement system employed, where **f** is the Fourier frequency. A commonly used frequency-domain measure is script L(f) = $\mathcal{L}$**(f),** the one-sided PSD of the phase fluctuations given by $S_\varphi(f)$, where φ is the residual phase deviation from a reference phase. Therefore, $\mathcal{L}$**(f)** = ½ $S_\varphi(f)$. One may derive

$$S_\varphi(f) = \frac{v_o^2}{f^2} S_y(f)$$ and $$S_\varphi(f) = (2\pi v_o)^2 S_x(f).$$ where x is the residual time deviation. Using the above equations, one may also write the following:

$$\sigma_y^2(\tau) = \frac{2}{(\pi v_o \tau)^2} \int_0^\infty S_\varphi(f) \, sin^4(\pi f \tau) \, df = \frac{8}{\tau^2} \int_0^\infty S_x(f) \, sin^4(\pi f \tau) \, df$$

Now the typical five power-law spectral density models often utilized in time and frequency metrology are $S_y(f) \sim f^\alpha;\ \alpha = -2, -1, 0, +1, +2$. PSDs are usually represented in a log-log plot, then the exponent α is the straight-line slope. As will be shown later, there are some simple equations that provide values for the frequency-domain measures as a function of the time-domain measures and vice-versa. However, in general, the time-domain measures cannot be integrated to obtain the frequency-domain measures. The above are the equations for AVAR. The following are the definitions and equations for the Modified Allan variance, MVAR, and for the Time variance, TVAR. The square roots of these three variances are commonly designated ADEV, MDEV, and TDEV, respectively.

For analyzing a time series of time deviations, x(t), (t = m τ_o with τ_o being the data acquisition spacing and m = 0, 1, 2, 3, etc. up to the data length, M) AVAR may be written as follows:

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \langle (\Delta^2 x)^2 \rangle ,$$

where the averaging time is τ. The Δ^2 is the second finite-difference operator -- operating on x. The angle brackets $\langle \rangle$ indicate that a time average is taken over the square of all the data available for this second-difference operating on x. For τ = nτ_o, the mth value of y is given by y_m = $(x_m - x_{m-n})/\tau$, and the mth second-finite difference of the time differences between the clocks for the time interval τ may be written $\Delta^2 x_m = x_{m+2n} - 2x_{m+n} - x_m$.

MVAR may be written as follows:

$$Mod.\,\sigma_y^2(\tau) = \frac{1}{2\tau^2}\langle(\Delta^2\bar{x})^2\rangle,$$

where the $\bar{x}$ is the time-difference residuals averaged over the interval τ. This averaging technique effectively modulates the bandwidth in the software allowing one to distinguish between white-noise phase modulation (PM) and flicker-noise PM.

For power-law spectral densities, AVAR and MVAR are proportional to τ^μ and to $\tau^{\mu'}$, respectively. From Lighthill's book on Generalized Functions... one can derive the following mapping:
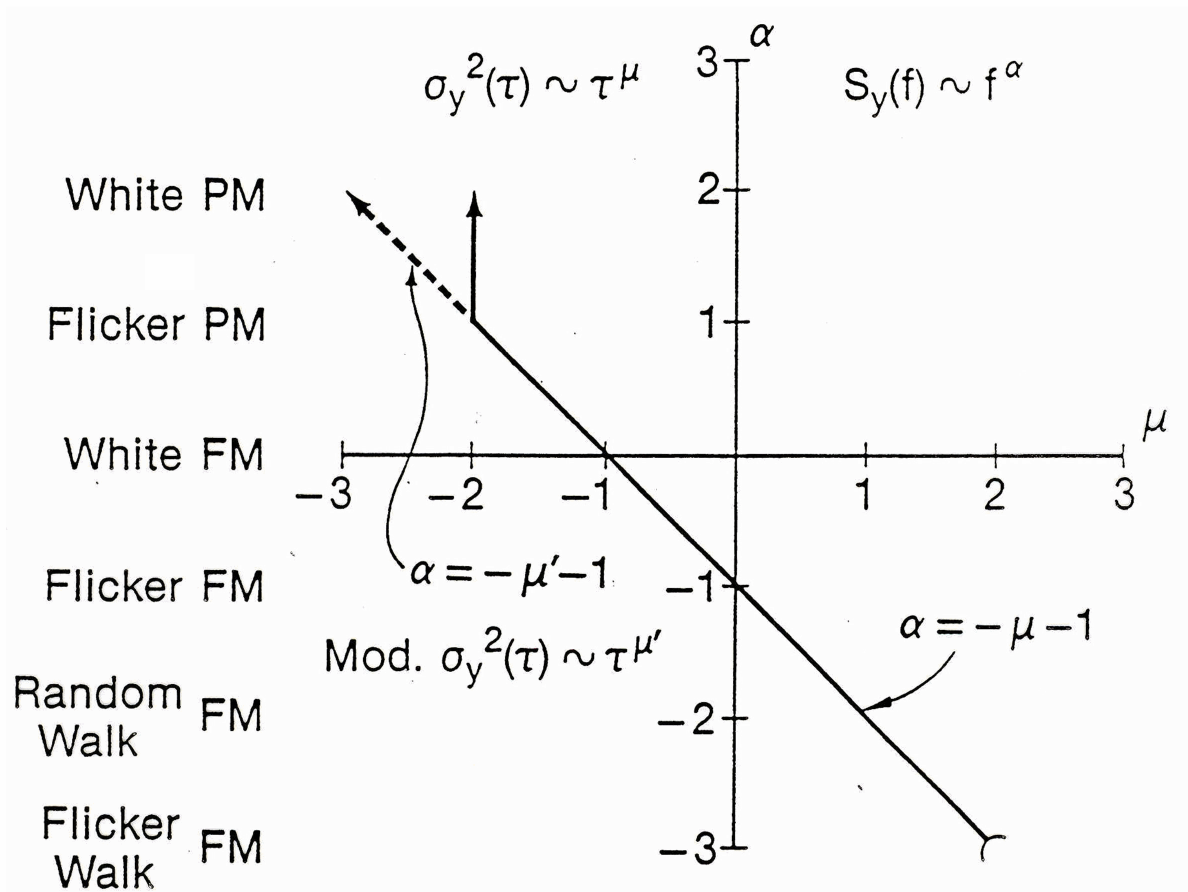
Finally, the Time Variance, TVAR, may be written as follows:

$$\sigma_x^2(\tau) = \frac{\tau^2}{3}\,Mod.\,\sigma_y^2(\tau) = \frac{1}{6}\langle(\Delta^2\bar{x})^2\rangle.$$

It is normalized to be equal to the classical variance for white-noise PM, while $\sigma_y^2(\tau)$ is normalized to be equal to the classical variance for white-noise FM.

For power-law spectra, TVAR has a similar proportionality relationship: $\sigma_x^2(\tau) \sim \tau^\eta$, where $\eta = -\beta - 1\ and\ \beta = \alpha - 2.$

Hence, as a log-log plot is made of these three variances versus the averaging time, τ, the slope indicates the kind of noise and the ordinate value gives the level of the noise.

The range of applicability for these three variances is given in the following chart:
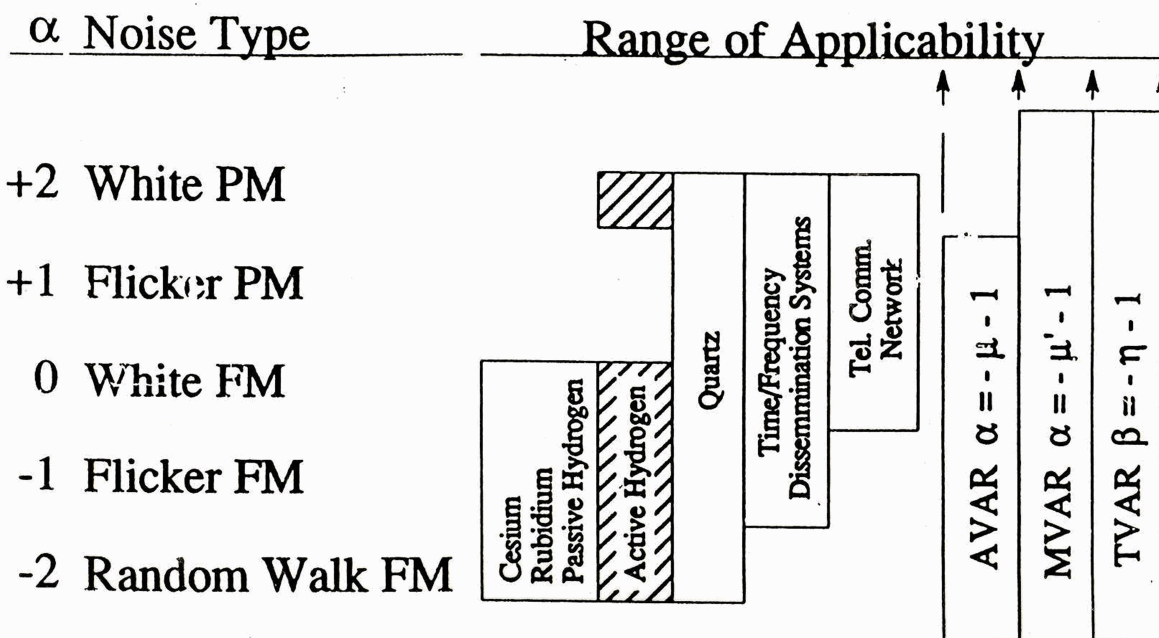
Now for each of these five noise types, the frequency-domain measures can be calculated from the time-domain measures by using the values taken from the following table. Of course, if the frequency deviations, phase deviations, or time deviations data are available then the spectral densities can be computed directly. If only the time-domain stability is known then this table will provide the answer, again, for these five power-law spectral types.

Note that when stability plots are made, if they are of $\mathscr{L}(f)$ or $S_\varphi(f)$ then they are dependent on the carrier frequency, ν_o, and its value needs to be indicated on the plot. For all of the other measures listed above (ADEV, MDEV, TDEV, $S_y(f)$, $S_x(f)$) the carrier frequency has been normalized out and need not be noted. White-noise PM and flicker-noise PM are also dependent on the measurement system bandwidth, f_h, and its value should be noted on the stability plots for all of the above measures when these models are used for the noise processes observed.

As a general note, these time-domain measures can be used in almost any area of metrology with significant benefits. Because of the general applicability, I wrote the following paper: *Should the Classical Variance be used as a Basic Measure in Standards Metrology?*
http://tf.boulder.nist.gov/general/pdf/776.pdf

NOISE TYPE	$S_y(f)$	$S_x(f)$
White PM	$\dfrac{(2\pi)^2}{3f_h} [\tau^2\sigma_y^2(\tau)] f^2$	$\dfrac{1}{\tau_0 f_h} [\tau\,\sigma_x^2(\tau)] f^0$
Flicker PM	$\dfrac{(2\pi)^2}{A} [\tau^2\sigma_y^2(\tau)] f^1$	$\dfrac{3}{3.37} [\tau^0\sigma_x^2(\tau)] f^{-1}$
White FM	$2 [\tau^1\sigma_y^2(\tau)] f^0$	$\dfrac{12}{(2\pi)^2} [\tau^{-1}\sigma_x^2(\tau)] f^{-2}$
Flicker FM	$\dfrac{1}{2\ell n2} [\tau^0\sigma_y^2(\tau)] f^{-1}$	$\dfrac{20}{(2\pi)^2 9\ell n2} [\tau^{-2}\sigma_x^2(\tau)] f^{-3}$
Random Walk FM	$\dfrac{6}{(2\pi)^2} [\tau^{-1}\sigma_y^2(\tau)] f^{-2}$	$\dfrac{240}{(2\pi)^4 11} [\tau^{-3}\sigma_x^2(\tau)] f^{-4}$

$$A = 1.038 + 3\ell n(2\pi f_h \tau)$$

A data set may need more than one spectral type or model to represent its performance. In that case, a spectral density needs to be computed for each spectral type, and then those different spectra can be added under the assumption of independence of the underlying driving force causing that type of spectral density, which is usually a good assumption, but not always.

Once either $S_y(f)$ or $S_x(f)$ is known, then the other spectral densities can be calculated using the equations above – including $\mathcal{L}$(f). As a time-domain to frequency-domain conversion example, suppose a quartz-crystal oscillator has a one-second stability $\sigma_y(\tau) = 2 \times 10^{-12}$ with a noise type of flicker FM, and we wish to know $\mathcal{L}$**(f=1Hz).** For this example, from the above

table and equations we may write the following for flicker-noise FM: $\mathcal{L}(f) = \dfrac{v_0^2}{4\ln2\,f^3}\,\sigma_y^2(\tau)$, which for this example is equal to 1.44 x 10^{-10} at 10 MHz. Expressed in dBc, this would be -98.4 dBc.

The confidence on the estimate of the above variances decreases as τ increases because of the decreasing number of degrees of freedom. The variances are Chi-squared distributed, and one can show for AVAR, specifically, for the longest possible averaging time (τ equal half the data length) the most probable value for AVAR is zero, and that these variances tend to be biased low for the largest averaging times available from a given length of data. Over the last couple of decades, David A. Howe in the Time and Frequency Division at NIST, Boulder and his team have done some remarkable work, and there are several publications, in removing these biases and increasing the effective degrees of freedom. Their work can be found in the Division's web site publications (search for Howe). http://tf.nist.gov/general/publications.htm

The frequency-domain to time-domain integral equations for all three variances is given as follows:

Translation Between Frequency and Time Domains

$$\tau = n\tau_0$$

AVAR:

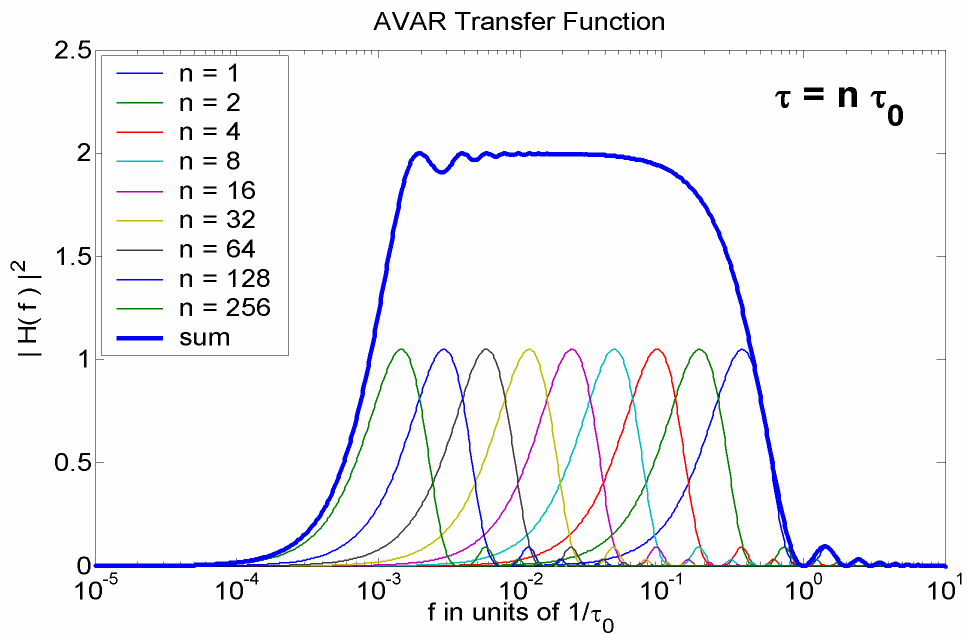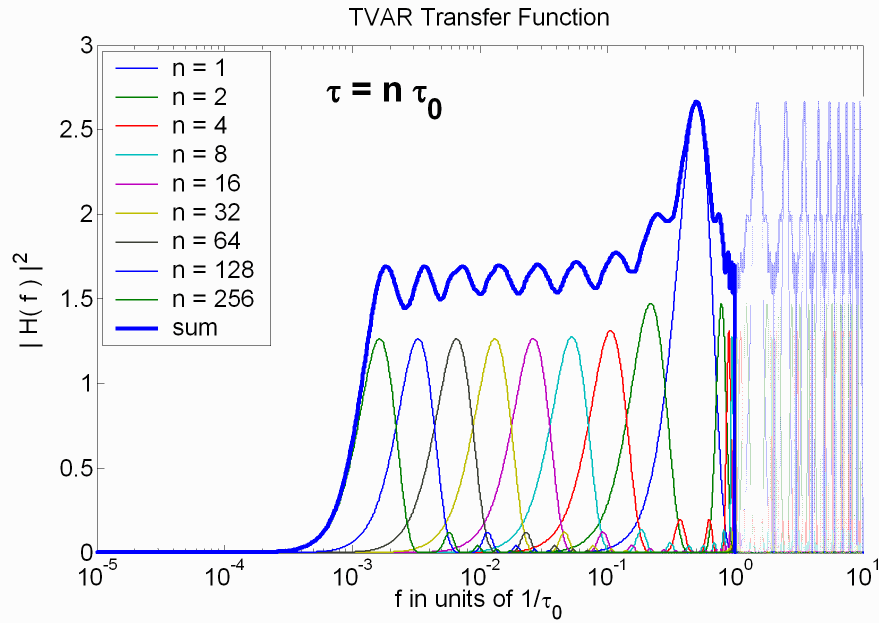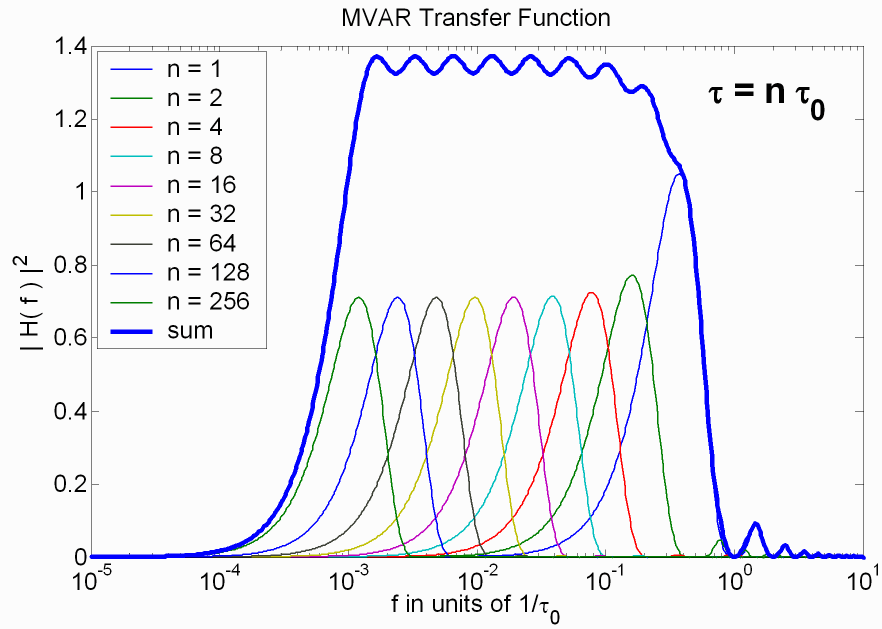$$\sigma_y^2(\tau) = \int_0^\infty 2\left[\frac{\sin^4(\pi f\tau)}{(\pi f\tau)^2}\right] S_y(f)df$$

MVAR:

$$\text{Mod}\,\sigma_y^2(\tau) = \int_0^\infty 2\left[\frac{\sin^3(\pi f\tau)}{(n\pi f\tau)\sin(\pi f\tau_0)}\right]^2 S_y(f)df$$

TVAR:

$$\sigma_x^2(\tau) = \frac{8}{3n^2}\int_0^\infty \left[\frac{\sin^3(\pi f\tau)}{\sin(\pi f\tau_0)}\right]^2 S_x(f)df$$

Typically, the values of τ are taken as a geometric progression: n = 1, 2, 4, 8, 16, etc. When that is done the following shows that these time-domain variances are nominally sampling a square window from the frequency domain.

MVAR Transfer Function

$\tau = n\,\tau_0$

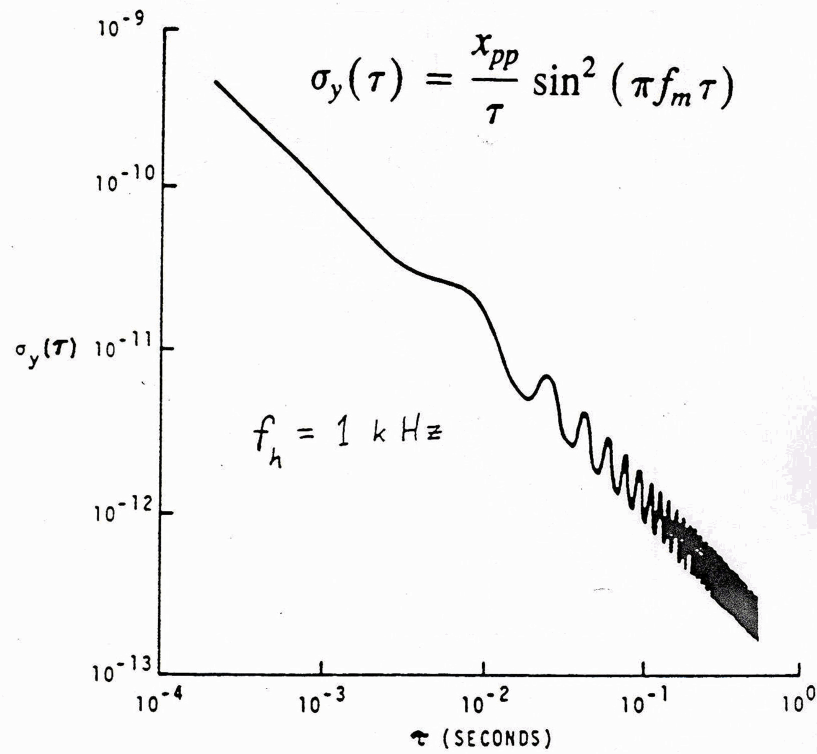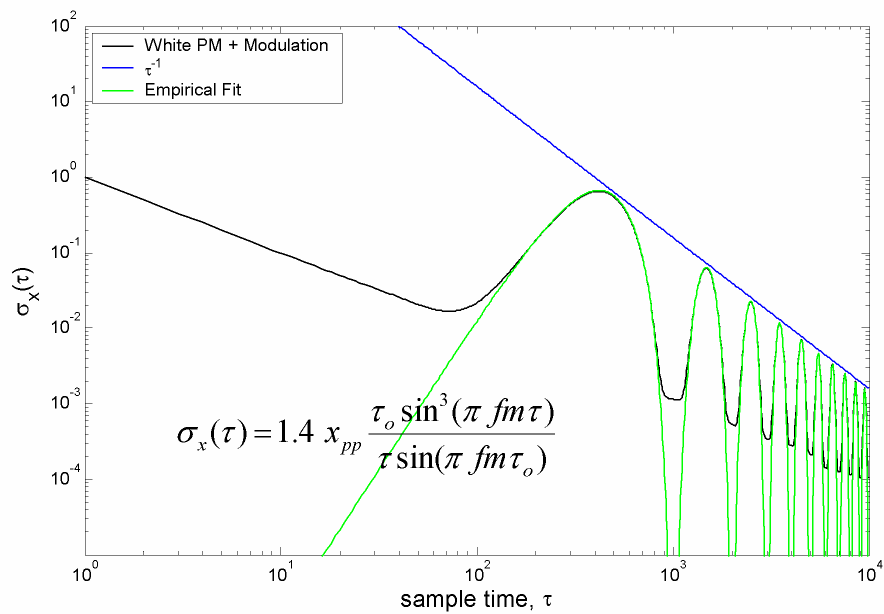TVAR Transfer Function

$\tau = n\,\tau_0$

Frequency-domain measures have a challenge in having high confidence on the low-Fourier frequency components. With the work of Howe, et. al. the time-domain can give high confidence estimates for large τ values (low-Fourier frequency components). When the data are available, it is typically recommended to analyze it both in the Frequency-domain as well as in the Time-domain. For example, bright spectral lines (side-bands) can best be observed in the Frequency-domain. A low-frequency side-band is best observed in the time-domain. It has an unusual water-fall signature as shown in the following for ADEV and TDEV, respectively.

EFFECTS OF SINGLE FREQUENCY
MODULATION ON $\sigma_y(\tau)$

$S_y(f) = 2.5 \times 10^{-15} [1 + 10^3 \, \delta(f - 60 \text{ Hz})]$

$$\sigma_y(\tau) = \frac{x_{pp}}{\tau} \sin^2(\pi f_m \tau)$$

$f_h = 1 \text{ kHz}$

$\sigma_y(\tau)$

τ (SECONDS)

Effect of fm on TDEV

White PM + Modulation
τ^{-1}
Empirical Fit

$\sigma_x(\tau)$

$$\sigma_x(\tau) = 1.4 \, x_{pp} \frac{\tau_o \sin^3(\pi \, fm \, \tau)}{\tau \sin(\pi \, fm \, \tau_o)}$$

sample time, τ

AVAR may be shown to be an optimum estimator for variations in frequency for atomic clocks. TVAR may be shown to be an optimum estimator for variations in time or phase, such as for measurement systems, telecommunication networks, etc. MVAR allows one to observe both clock and measurement instabilities in a near optimum way and it removes the ambiguity problem existing for AVAR for power-law spectral density values where $\alpha \geq 1$. One may also show that the time prediction error of a clock, $x_p(\tau)$, over an prediction interval τ_p equal to τ , under the assumption of optimum prediction, is given by the following for the various power-law spectra. So if ADEV is known over various τ values one can calculate the time keeping ability for that clock. This has been done for a big variety of clocks as shown below.

Clock-time Keeping Ability

$$x_p(\tau) = k \ \tau \ \sigma_y(\tau)$$

$$k = 1 \ \textit{for white FM and random walk FM}$$

$$k = 1/\sqrt{\ln 2} \ = 1.2 \ \textit{for flic} \mathrm{ker} FM$$

$$k = 1/\sqrt{3} \ \textit{for white PM and flic} \mathrm{ker} PM$$

It is interesting that the time dispersion of the earth's spin rate is the largest over very long intervals. It is also of interest to note that over very large intervals the earth's orbit stability may be better than that of atomic clocks. Not enough data exist to show that either way. Current atomic clocks are by far the most accurate measurements known with uncertainties being analogous to less than one second in 3.7 billion years:

http://www.nist.gov/pml/div688/

One will notice that the short-term stability of quartz-crystal oscillators is outstanding. This has been utilized in an ensemble mode to obtain atomic-clock like frequency stabilities for even longer averaging times. This technology is called EQUATE and has the ability to also sense rotation and translation. The above variances are highly utilized in the development of this technology with the goal of providing excellent timing and navigation and removing coordinate drift. If this can be done it will be unique and very useful in a large variety of applications.
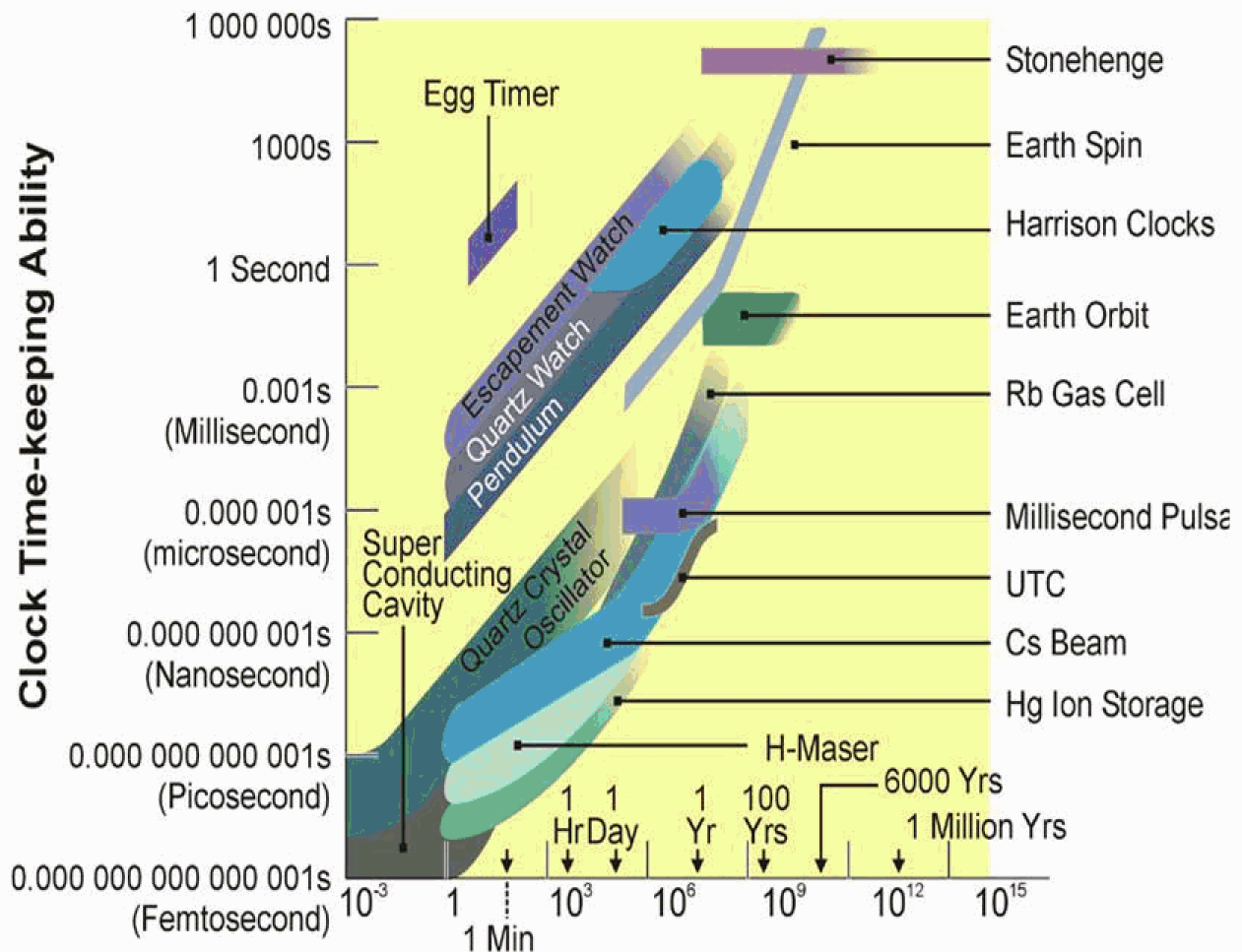
Time Since Synchronization (s)

Suggested References and References used:

http://tf.nist.gov/general/publications.htm

NIST Technical Note 1337, *Characterization of Clocks and Oscillators* [1990]

http://www.allanstime.com/Publications/DWA/Science_Timekeeping/index.html

ITU HANDBOOK: *Selection and Use of Precise Frequency and Time Systems*

American Association of Physics Teachers: *Time and Frequency Measurement*